

Isometries of a D3-brane space

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We obtain the Killing equations and the corresponding infinitesimal isometries for the ten dimensional space generated by a large number of coincident D3-branes. In a convenient limit this space becomes an $AdS_5 \times S^5$ which is relevant for the AdS/CFT correspondence. In this case, using Poincaré coordinates, we also write down the Killing equations and infinitesimal isometries. Then we obtain a simple realization of the isomorphism between AdS isometries and the boundary conformal group.

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I. INTRODUCTION

In the supergravity approximation for low energy string theory one finds non trivial background solutions known as Dp-branes[1]. These objects were later identified as solitons in string theory[2]. The ten dimensional geometry generated by a large number of coincident D3-branes has a limit corresponding to an $AdS_5 \times S^5$ space. This limit is an essential ingredient for the duality found by Maldacena between string theory in AdS spaces and supersymmetric conformal field theories on their boundaries[3] (AdS/CFT correspondence). Prescriptions detailing this correspondence were presented in [4, 5] where it was shown how to calculate boundary correlation functions from the AdS bulk string theory (for a review and a wide list of references see[6]). Furthermore it was shortly pointed out [5] that this duality can be understood as a realization of the holographic principle[7, 8, 9]. This principle states that in a quantum theory with gravity (as is the case of string theory) all the information contained in some spatial region can be mapped on a corresponding boundary. This principle

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was inspired by the study of quantum aspects of black hole entropy[10, 11].

In this article we are going to study the isometries of the ten dimensional space generated by D3-branes. We will also consider the Killing equations and the corresponding isometries in the $AdS_5 \times S^5$ limit. As it is well known AdS_{n+1} space is maximally symmetric with isometry group $SO(2, n)$ which is isomorphic to the conformal group defined in an n dimensional flat space. Here we obtain a simple explicit realization of this isomorphism using Poincaré coordinates. These coordinates are important in the study of the AdS/CFT correspondence, as for example in calculating boundary correlation functions. Previous studies on the isometries of AdS or asymptotically AdS spaces include [4, 12, 13, 14, 15]. Also some aspects of the compactification of AdS space and its relation to the D3-brane space were discussed in [16].

For a general space with coordinates x^μ , infinitesimal isometries are defined by the condition of invariance of the metric $g_{\mu\nu}$ under the transformations[17]

$$x'^\mu = x^\mu + \epsilon \xi^\mu, \quad (1)$$

where ϵ is an arbitrary infinitesimal parameter. This implies the Killing equations

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \quad (2)$$

or explicitly

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\Gamma_{\mu\nu}^\sigma \xi_\sigma = 0, \quad (3)$$

where the Christoffel symbols are given by

$$\Gamma_{\mu\nu}^\sigma = \frac{1}{2} g^{\sigma\rho} \left[\frac{\partial g_{\mu\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right].$$

II. D3-BRANE SPACE ISOMETRIES

The invariant measure $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ of the ten dimensional geometry generated by a large number N of coincident D3-branes can be written as[1, 4]

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} (-dt^2 + d\vec{x}^2) + \left(1 + \frac{R^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2) \quad (4)$$

where $R^4 = N/2\pi^2 T_3$ and T_3 is the tension of a single D3-brane. Changing the axial coordinate according to: $z = R^2/r$, the metric takes the form:

$$ds^2 = \frac{R^2}{z^2} \sqrt{f(z)} \left(-dt^2 + d\vec{x}^2 \right) + \frac{R^2}{\sqrt{f(z)}} \left(\frac{dz^2}{z^2} + d\Omega_5^2 \right), \quad (5)$$

where

$$f(z) = \frac{z^4}{z^4 + R^4}.$$

From now on we take the Euclidean version of this metric so that $(t, \vec{x}) \equiv x^i$, $i = 1, \dots, 4$. The coordinates of $d\Omega_5$ are represented as θ^α with $\alpha = 1, \dots, 5$, assumed to be orthogonal: $d\Omega_5^2 = \tilde{g}_{\alpha\alpha}(d\theta^\alpha)^2$. This implies the following non vanishing z dependent Christoffel symbols

$$\Gamma_{00}^0 = \frac{f(z) - 2}{z} \quad (6)$$

$$\Gamma_{0j}^i = -\frac{f(z)}{z} \delta_j^i \quad (7)$$

$$\Gamma_{ij}^0 = \frac{[f(z)]^2}{z} \bar{\delta}_{ij} \quad (8)$$

$$\Gamma_{\alpha\beta}^0 = \tilde{g}_{[\alpha\alpha]}(\theta) z (1 - f(z)) \bar{\delta}_{\alpha\beta} \quad (9)$$

$$\Gamma_{0\beta}^\alpha = \frac{f(z) - 1}{z} \delta_\beta^\alpha. \quad (10)$$

Our notation is: $z \equiv x^0$, Latin indices (i, j, k, l, \dots) correspond to the variables x^i and Greek indices $(\alpha, \beta, \gamma, \dots)$ to the angular variables of $d\Omega_5$. Note that $\Gamma_{\alpha\beta}^\gamma$ are in general non vanishing but are independent of z (and x^i). The subscript $[\alpha]$ means that we are not summing over this index. Note also that $\delta_j^i, \delta_\beta^\alpha$ are the usual Kronecker tensors and we are defining the symbols $\bar{\delta}_{ij}$ and $\bar{\delta}_{\alpha\beta}$ to be one when their indices are equal and zero otherwise, so that they are not tensors in this curved space-time.

For this D3-brane space the Killing equations (3) become

$$\partial_i \xi_j + \partial_j \xi_i - \frac{2[f(z)]^2}{z} \bar{\delta}_{ij} \xi_0 = 0 \quad (11)$$

$$\partial_0 \xi_i + \partial_i \xi_0 + \frac{2f(z)}{z} \xi_i = 0 \quad (12)$$

$$\partial_0 \xi_0 + \frac{2 - f(z)}{z} \xi_0 = 0 \quad (13)$$

$$\partial_i \xi_\alpha + \partial_\alpha \xi_i = 0 \quad (14)$$

$$\partial_0 \xi_\alpha + \partial_\alpha \xi_0 + 2 \frac{1 - f(z)}{z} \xi_\alpha = 0 \quad (15)$$

$$\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha + 2z(f(z) - 1) \bar{\delta}_{\alpha\beta} F_{[\alpha]}(\theta) \xi_0 - 2\Gamma_{\alpha\beta}^\gamma \xi_\gamma = 0. \quad (16)$$

It is simpler to solve the corresponding equations for the contravariant Killing vectors that

read

$$\bar{\delta}_{jk}\partial_i \xi^k + \bar{\delta}_{ik}\partial_j \xi^k - \frac{2f(z)}{z} \bar{\delta}_{ij} \xi^0 = 0 \quad (17)$$

$$\bar{\delta}_{ik}\partial_0 \xi^k + \frac{1}{f(z)}\partial_i \xi^0 = 0 \quad (18)$$

$$\partial_0 \xi^0 + \frac{f(z)-2}{z} \xi^0 = 0 \quad (19)$$

$$\bar{\delta}_{ik} \partial_\alpha \xi^k + \frac{z^2}{f(z)} \tilde{g}_{\alpha\beta} \partial_i \xi^\beta = 0 \quad (20)$$

$$\partial_\alpha \xi^0 + z^2 \tilde{g}_{\alpha\beta} \partial_0 \xi^\beta = 0 \quad (21)$$

$$\partial_\sigma \tilde{g}_{\alpha\beta} \xi^\sigma + \tilde{g}_{\beta\sigma} \partial_\alpha \xi^\sigma + \tilde{g}_{\alpha\sigma} \partial_\beta \xi^\sigma + 2 \frac{(f(z)-1)}{z} \bar{\delta}_{\alpha\beta} \tilde{g}_{[\alpha\gamma]}(\theta) \xi^0 = 0. \quad (22)$$

Equation (19) constrains the dependence of ξ^0 on the coordinate z to the form

$$\xi^0 = z [f(z)]^{1/4} G(x^i, \theta^\alpha) \quad (23)$$

where $G(x^i, \theta^\alpha)$ is some function to be determined. Now, equation (17) takes the form

$$\bar{\delta}_{jk}\partial_i \xi^k + \bar{\delta}_{ik}\partial_j \xi^k = 2 \bar{\delta}_{ij} [f(z)]^{5/4} G(x^i, \theta^\alpha) \quad (24)$$

Taking $i = j$ in the above equation we obtain $[f(z)]^{5/4} G(x^i, \theta^\alpha) = (1/d) \partial_k \xi^k$ where $d = 4$ is the dimension of the space spanned by the coordinates x^i . Inserting this back into eq. (24) we get

$$\bar{\delta}_{jk}\partial_i \xi^k + \bar{\delta}_{ik}\partial_j \xi^k - \frac{1}{2} \bar{\delta}_{ij} \partial_k \xi^k = 0 \quad (25)$$

which is the conformal group equation (for coordinates x^i). From this equation we get:

$$\left[\bar{\delta}_{ij} \partial_k \partial_k + 2 \partial_i \partial_j \right] \partial_l \xi^l = 0. \quad (26)$$

This tells us that all the second derivatives of $\partial_l \xi^l$ with respect to the x^i variables vanish. This determines the general quadratic form of ξ^i in the coordinates x^j , as usual in the conformal group transformations. Then taking into account equation (24) we find

$$\xi^i = x^k \omega^{ik}(\theta^\alpha, z) + a^i(\theta^\alpha, z) + \left[x^i \lambda(\theta^\alpha) + x^j x^j d^i(\theta^\alpha) - 2x^i x^j d^j(\theta^\alpha) \right] \frac{z^5}{(R^4 + z^4)^{5/4}} \quad (27)$$

where $\omega^{ik}(\theta^\alpha, z) = -\omega^{ki}(\theta^\alpha, z)$. Substituting this result in eq. (24) we find

$$G(x^i, \theta^\alpha) = \lambda(\theta^\alpha) - 2x^j d^j(\theta^\alpha).$$

Now imposing equation (18) we find

$$\begin{aligned} x^k \partial_0 \omega^{ik}(\theta^\alpha, z) + \partial_0 a^i(\theta^\alpha, z) + \left[x^i \lambda(\theta^\alpha) + x^j x^j d^i(\theta^\alpha) - 2x^i x^j d^j(\theta^\alpha) \right] \partial_0 \left(\frac{z^5}{(R^4 + z^4)^{5/4}} \right) \\ = 2z[f(z)]^{-3/4} d^i(\theta^\alpha) \end{aligned} \quad (28)$$

Comparing the terms with the same power of z and x^i we conclude that

$$\lambda = 0$$

$$\partial_0 \omega^{ik} = 0$$

$$d^i = 0$$

$$\partial_0 a^i = 0 ,$$

so that $G(x^i, \theta^\alpha) = 0$. Then from eqs. (23) and (27) we find

$$\xi^0 = 0 \quad (29)$$

$$\xi^i = x^k \omega^{ik}(\theta^\alpha) + a^i(\theta^\alpha) . \quad (30)$$

Using the result $\xi^0 = 0$ in eq. (21) we get the condition:

$$\partial_0 \xi^\alpha = 0 \quad (31)$$

Now, since both ξ^α and ξ^i are independent of the coordinate z , equation (20) can only be satisfied if both terms vanish independently. This implies that: (i) ω^{ik} and a^i are also independent of coordinates θ^α , that means they are *constants*; (ii) ξ^α are also independent of coordinates x^i .

So, the isometries of the ten dimensional D3-brane system can finally be written as

$$\xi^0 = 0 \quad (32)$$

$$\xi^i = a^i + \omega^{ij} x^j \quad (33)$$

$$\xi^\alpha = \xi^\alpha(\theta) . \quad (34)$$

where $\xi^\alpha(\theta) \equiv \xi^\alpha(\theta^1, \dots, \theta^5)$ represent the usual isometries of S^5 .

These solutions show that this space is not invariant for transformations in the coordinate z . This could be expected due to the presence of different factors of z in the D3-brane metric (5). Note that the isometries in the x^i coordinates correspond to ten independent parameters a^i and ω^{ij} . These isometries are isomorphic to Poincaré transformations in 4 dimensional flat space. Furthermore there is an S^5 invariance in the θ^α coordinates with 15 independent parameters since S^n spaces are maximally symmetric.

III. ADS LIMIT

In the limit $z \gg R$ the D3-brane metric eq. (5) takes the form of an $AdS_5 \times S^5$ space

$$ds^2 = \frac{R^2}{z^2} (dz^2 + (dx^i)^2) + R^2 d\Omega_5^2, \quad (35)$$

where the AdS_5 space with radius R is represented by Poincaré coordinates (z, x^i) with $i = 1, \dots, 4$. For this metric the Christoffel symbols are

$$\Gamma_{00}^0 = -\frac{1}{z} \quad (36)$$

$$\Gamma_{0i}^j = -\frac{1}{z} \delta_i^j \quad (37)$$

$$\Gamma_{ij}^0 = -\frac{1}{z} \bar{\delta}_{ij} \quad (38)$$

$$\Gamma_{\alpha\beta}^0 = 0 = \Gamma_{0\alpha}^\beta \quad (39)$$

and $\Gamma_{\alpha\beta}^\gamma$ are independent of z and x^i , as in the D3-brane space. Note that the symbols (36)-(39) can also be obtained from the ones in the brane system eqs. (6)-(10) considering the limit $z \gg R$ such that $f(z) \cong 1 - R^4/z^4$.

The Killing equations now take the form

$$\bar{\delta}_{jk} \partial_i \xi^k + \bar{\delta}_{ik} \partial_j \xi^k - \frac{2}{z} \bar{\delta}_{ij} \xi^0 = 0 \quad (40)$$

$$\bar{\delta}_{ik} \partial_0 \xi^k + \partial_i \xi^0 = 0 \quad (41)$$

$$\partial_0 \xi^0 - \frac{1}{z} \xi^0 = 0 \quad (42)$$

$$\bar{\delta}_{ik} \partial_\alpha \xi^k + z^2 \tilde{g}_{\alpha\beta} \partial_i \xi^\beta = 0 \quad (43)$$

$$\partial_\alpha \xi^0 + z^2 \tilde{g}_{\alpha\beta} \partial_0 \xi^\beta = 0 \quad (44)$$

$$\partial_\sigma \tilde{g}_{\alpha\beta} \xi^\sigma + \tilde{g}_{\beta\sigma} \partial_\alpha \xi^\sigma + \tilde{g}_{\alpha\sigma} \partial_\beta \xi^\sigma = 0. \quad (45)$$

We follow the same procedure as in the previous section to find the isometries in this case.

From equation (42) we find

$$\xi^0 = z G(x^i, \theta^\alpha) \quad (46)$$

In this case the solution for eq. (40) reads

$$\xi^i = x^k \omega^{ik}(\theta^\alpha, z) + \tilde{a}^i(\theta^\alpha, z) + x^i \lambda(\theta^\alpha) + x^j x^j d^i(\theta^\alpha) - 2x^i x^j d^j(\theta^\alpha) \quad (47)$$

where again $\omega^{ik}(\theta^\alpha, z) = -\omega^{ki}(\theta^\alpha, z)$ and $G(x^i, \theta^\alpha) = \lambda(\theta^\alpha) - 2x^j d^j(\theta^\alpha)$ as in the D3-brane case.

Now imposing equation (41) we find

$$x^k \partial_0 \omega^{ik}(\theta^\alpha, z) + \partial_0 \tilde{a}^i(\theta^\alpha, z) = 2z d^i(\theta^\alpha). \quad (48)$$

Note that this equation, in contrast to the corresponding D3-brane equation (28), does not imply the vanishing of λ and d^i (and consequently ξ^0) but only

$$\partial_0 \omega^{ik}(\theta^\alpha, z) = 0$$

$$\partial_0 \tilde{a}^i(\theta^\alpha, z) = 2z d^i(\theta^\alpha),$$

that means $\omega^{ik} = \omega^{ik}(\theta^\alpha)$ and $\tilde{a}^i(\theta^\alpha, z) = z^2 d^i(\theta^\alpha) + a^i(\theta^\alpha)$.

Then from eqs. (43) and (44) we see that ω^{ik} , d^i , a^i and λ do not depend on θ^α . Furthermore ξ^α can only depend on the θ coordinates. So the isometries in this case finally read

$$\xi^0 = (\lambda - 2d^i x^i) z \quad (49)$$

$$\xi^i = a^i + \omega^{ij} x^j + \lambda x^i + d^i x^j x^j - 2x^i x^j d^j + z^2 d^i \quad (50)$$

$$\xi^\alpha = \xi^\alpha(\theta). \quad (51)$$

The 15 independent parameters a^i , ω^{ij} , d^i , λ of eqs. (49) and (50) represent the invariances of the AdS_5 space. This enlarges the isometries with respect to the D3-brane space. This happens due to the non vanishing ξ^0 in opposition to the previous case. The S^5 isometries represented by (51) are the same as in the D3-brane case. Note that the isometries of the AdS_5 and S^5 are independent as expected. Furthermore, the AdS_5 and S^5 spaces are both maximally symmetric so the isometry group of metric (35) has 30 independent parameters.

Using these results, we now analyze the relation between the AdS isometries and the induced isometries on surfaces of constant z . First we consider a surface $z = \text{constant} \neq 0$. The subgroup of isometries in this case corresponds to those with $\xi^0 = 0$ which implies $\lambda = 0$ and $d^i = 0$. Then in this case we find Poincaré invariance

$$\xi^i = a^i + \omega^{ij} x^j \quad (52)$$

in such a surface. Additionally these surfaces also have S^5 isometries. So we see that the isometries of $AdS_5 \times S^5$ at a fixed $z \neq 0$ correspond exactly to the ones of the ten dimensional D3-brane system.

The very important particular case of the surface $z = 0$ corresponds to the *AdS* boundary. On this surface the condition $\xi^0 = 0$ does not imply the vanishing of λ and d^i , as can be seen from equation (49). These parameters remain arbitrary so the isometries on the boundary are

$$\xi^i = a^i + \omega^{ij}x^j + \lambda x^i + d^i x^j x^j - 2x^i x^j d^j . \quad (53)$$

These isometries correspond to infinitesimal conformal transformations on the x^i coordinates.

Let us now comment on the relations between the isometry subgroups in the bulk and on the boundary. First note that dilatations on the boundary, defined by $\lambda \neq 0$ and the other parameters vanishing in eq. (53) correspond to dilatations in the five dimensional *AdS* space coordinates given by eqs.(49),(50) with the same choice of parameters. Second, translations and rotations on the boundary given by non vanishing a^i and ω^{ij} imply the same transformations in the *AdS* bulk with fixed value of the z coordinate. Finally special conformal transformations on the boundary given by non vanishing d^i correspond to the bulk isometries

$$\xi^0 = -2d^i x^i z \quad (54)$$

$$\xi^i = d^i (x^j x^j + z^2) - 2x^i x^j d^j . \quad (55)$$

This represents a special conformal transformation in the coordinates $x^\mu = (z, x^i)$ with parameter $d^\mu = (0, d^i)$.

In summary, we found the isometries of the D3-brane space-time. Then we considered the *AdS* limit where we have shown explicitly that the isometry group acts on the boundary as the conformal group. This result is a simple realization in Poincaré coordinates of the isomorphism between these two groups. This is an essential ingredient for the *AdS/CFT* correspondence.

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- [1] G. Horowitz and A. Strominger, Nucl. Phys. B360 (1991) 197.
 - [2] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724.
 - [3] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231.
 - [4] S. S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys. Lett. B428 (1998) 105.
 - [5] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253.
 - [6] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323 (2000) 183.
 - [7] G. 't Hooft, "Dimensional reduction in quantum gravity" in Salam Festschrift, eds. A. Aly, J. Ellis and S. Randjbar-Daemi, World Scientific, Singapore, 1993, gr-qc/9310026.
 - [8] L. Susskind, J. Math. Phys. 36 (1995) 6377.
 - [9] L. Susskind and E. Witten, "The holographic bound in anti-de Sitter space", SU-ITP-98-39, IASSNS-HEP-98-44, hep-th/9805114.
 - [10] J. D. Bekenstein, Lett. Nuov. Cim. 4 (1972) 737, Phys. Rev. D7 (1973) 2333.
 - [11] S. W. Hawking, Phys. Rev. D13 (1976) 2460.
 - [12] J.D. Brown, M. Henneaux, Commun. Math. Phys. 104 (1986) 207.
 - [13] J. L. Petersen, Int. J. Mod. Phys. A14 (1999) 3597.
 - [14] G. Barnich, F. Brandt, Nucl. Phys. B633 (2002) 3.
 - [15] G. Barnich, F. Brandt, K. Claes, Nucl. Phys. B (Proc. Suppl.) 127 (2004) 114.
 - [16] H. Boschi-Filho and N. R. F. Braga, Phys. Rev. D66 (2002) 025005.
 - [17] See for instance: M. Carmeli, Classical Fields: General Relativity and Gauge Theory, Wiley, New York, 1982.